

**Paper 3A: Further Pure Mathematics 1 Mark Scheme**

| Question   | Scheme  |            |                |       |                 |             |                 | Marks       | AOs  |      |
|--|---|------------|----------------|-------|-----------------|-------------|-----------------|-------------|------|------|
| <b>1</b>   | Step 0.5  |            |                |       |                 |             |                 | B1          | 1.1b |      |
|  |   | $y_0$      | $y_1$          | $y_2$ | $y_3$           | $y_4$       | $y_5$           | $y_6$       | M1   | 1.1b |
|  | $x$   | 1          | 1.5            | 2     | 2.5             | 3           | 3.5             | 4           |      |      |
|  | $y$   | $\sqrt{2}$ | $\sqrt{4.375}$ | 3     | $\sqrt{16.625}$ | $\sqrt{28}$ | $\sqrt{43.875}$ | $\sqrt{65}$ |      |      |
|  | $y_0 + 4y_1 + 2y_2 + 4y_3 + 2y_4 + 4y_5 + y_6 = "77.23"$        |            |                |       |                 |             |                 | M1          | 1.1b |      |
|  | $\int_1^4 \sqrt{1+x^3} dx \approx \frac{0.5}{3} \times "77.23"$ |            |                |       |                 |             |                 | M1          | 1.1b |      |
|  | $= 12.9$  |            |                |       |                 |             |                 | A1          | 1.1b |      |
|  |   |            |                |       |                 |             | <b>(5)</b>      |             |      |      |
| <b>(5 marks)</b>   |   |            |                |       |                 |             |                 |             |      |      |
| <b>Notes:</b>  |   |            |                |       |                 |             |                 |             |      |      |
| <b>B1:</b> Use of step length 0.5  |   |            |                |       |                 |             |                 |             |      |      |
| <b>M1:</b> Attempt to find $y$ values with at least 2 correct  |   |            |                |       |                 |             |                 |             |      |      |
| <b>M1:</b> Use of formula " $y_0 + 4y_1 + 2y_2 + 4y_3 + 2y_4 + 4y_5 + y_6$ " with correct coefficients |   |            |                |       |                 |             |                 |             |      |      |
| <b>A1:</b> $\frac{0.5}{3} \times$ their 77.23  |   |            |                |       |                 |             |                 |             |      |      |
| <b>A1:</b> awrt 12.9   |   |            |                |       |                 |             |                 |             |      |      |

| Question   | Scheme  | Marks | AOs  |
|--|---|-------|------|
| 2  | $y = x^3 e^{kx}$ so $u = x^3$ and<br>$\frac{du}{dx} = 3x^2$ and $\frac{d^2u}{dx^2} = 6x$ and $\frac{d^3u}{dx^3} = 6$ (and $\frac{d^4u}{dx^4} = 0$ )                         | M1    | 1.1b |
|  | $v = e^{kx}$ and $\frac{d^n v}{dx^n} = k^n e^{kx}$ and $\frac{d^{n-1} v}{dx^{n-1}} = k^{n-1} e^{kx}$ and $\frac{d^{n-2} v}{dx^{n-2}} = k^{n-2} e^{kx}$<br>(and...)          | M1    | 2.1  |
|  | $\frac{d^n y}{dx^n} = x^3 k^n e^{kx} + n3x^2 k^{n-1} e^{kx} + \frac{n(n-1)}{2} 6x k^{n-2} e^{kx} + \frac{n(n-1)(n-2)}{3!} 6k^{n-3} e^{kx}$<br>and remaining terms disappear | M1    | 2.1  |
|  | So $\frac{d^n y}{dx^n} = k^{n-3} e^{kx} (k^3 x^3 + 3nk^2 x^2 + 3n(n-1)kx + n(n-1)(n-2))$ *  | A1*   | 1.1b |
|  |   | (4)   |      |
| <b>(4 marks)</b>   |   |       |      |
| <b>Notes:</b>  |   |       |      |
| <b>M1:</b> Differentiate $u = x^3$ three times<br><b>M1:</b> Use $u = e^{kx}$ and establish the form of the derivatives, with at least the three shown<br><b>M1:</b> Uses correct formula, with 2 and 3! (or 6) and with terms shown to disappear after the fourth term<br><b>A1*:</b> Correct solution leading to the given answer stated. No errors seen |   |       |      |

| Question          | Scheme   | Marks      | AOs  |
|-------------------|--|------------|------|
| <b>3(a)</b>       | Use of $x = tv$ to give $\frac{dx}{dt} = v + t \frac{dv}{dt}$  | M1         | 1.1b |
|                   | Hence $\frac{d^2x}{dt^2} = \frac{dv}{dt} + \frac{dv}{dt} + t \frac{d^2v}{dt^2}$  | M1         | 2.1  |
|                   |  | A1         | 1.1b |
|                   | Uses $t^2$ (their 2 <sup>nd</sup> derivative) $- 2t$ (their 1 <sup>st</sup> derivative) $+ (2 + t^2)x = t^4$ and simplifies LHS  | M1         | 2.1  |
|                   | $\left( t^3 \frac{d^2v}{dt^2} + t^3v = t^4 \text{ leading to} \right) \frac{d^2v}{dt^2} + v = t^*$   | A1*        | 1.1b |
|                   | <b>(5)</b>   |            |      |
| <b>(b)</b>        | Solve $\lambda^2 + 1 = 0$ to give $\lambda^2 = -1$   | M1         | 1.1b |
|                   | $v = A \cos t + B \sin t$  | A1ft       | 1.1b |
|                   | Particular Integral is $v = kt + l$  | B1         | 2.2a |
|                   | $\frac{dv}{dt} = k$ and $\frac{d^2v}{dt^2} = 0$ and solve $0 + kt + l = t$ to give $k = 1, l = 0$  | M1         | 1.1b |
|                   | Solution: $v = A \cos t + B \sin t + t$  | A1         | 1.1b |
|                   | Displacement of $C$ from $O$ is given by $x = tv = \dots$  | M1         | 3.4  |
|                   | $x = t(A \cos t + B \sin t + t)$   | A1         | 2.2a |
|                   | <b>(7)</b>   |            |      |
| <b>(c)(i)</b>     | For large $t$ , the displacement gets very large (and positive)  | B1         | 3.2a |
| <b>(ii)</b>       | Model suggests midpoint of spring moving relative to fixed point has large displacement when $t$ is large, which is unrealistic. The spring may reach elastic limit / will break | B1         | 3.5a |
|                   |  | <b>(2)</b> |      |
| <b>(14 marks)</b> |  |            |      |

|                          |   |
|--------------------------|---|
| <b>Question 3 notes:</b> |   |
| <b>(a)</b>               |   |
| <b>M1:</b>               | Uses product rule to obtain first derivative  |
| <b>M1:</b>               | Continues to differentiate again, with product rule and chain rule as appropriate, in order to establish the second derivative        |
| <b>A1:</b>               | Correct second derivative. Accept equivalent expressions  |
| <b>M1:</b>               | Shows clearly the substitution into the given equation in order to form the new equation and gathers like terms                       |
| <b>A1*:</b>              | Fully correct solution leading to the given answer  |
| <b>(b)</b>               | Accept variations on symbols for constants throughout   |
| <b>M1:</b>               | Form and solve a quadratic Auxiliary Equation   |
| <b>A1ft:</b>             | Correct form of the Complementary Function for their solutions to the AE  |
| <b>B1:</b>               | Deduces the correct form for the Particular Integral (note $v = mt^2 + kt + l$ is fine)   |
| <b>M1:</b>               | Differentiates their Particular Integral and substitutes their derivatives into the equation to find the constants ( $m = 0$ if used) |
| <b>A1:</b>               | Correct general solution for equation (II)  |
| <b>M1:</b>               | Links the solution to equation (II) to the solution of the model equation correctly to find the displacement equation                 |
| <b>A1:</b>               | Deduces the correct general solution for the displacement   |
| <b>(c)(i)</b>            |   |
| <b>B1:</b>               | States that for large $t$ the displacement is large o.e. Accept e.g. as $t \rightarrow \infty$ , $x \rightarrow \infty$               |
| <b>(c)(ii)</b>           |   |
| <b>B1:</b>               | Reflect on the context of the original problem. Accept 'model unrealistic' / 'spring will break'                                      |

| Question   | Scheme  | Marks      | AOs  |
|--|---|------------|------|
| <b>4(a)</b>  | $y'' = 2xy' - y \Rightarrow y''' = 2xy'' + 2y' - y''$   | M1         | 1.1b |
|  |   | A1         | 1.1b |
|  | $y''' = 2xy'' + y' \Rightarrow y'''' = 2xy''' + 2y'' + y''$   | M1         | 2.1  |
|  | $y'''' = 2xy'''' + 3y'' \Rightarrow y'''''' = 2xy'''' + 5y''''$   | A1         | 2.1  |
|  |   | <b>(4)</b> |      |
| <b>(b)</b>   | $x = 0, y = 0, y' = 1 \Rightarrow y''(0) = 0\pi$ from equation  | B1         | 2.2a |
|  | $y'''(0) = 2 \times 0 \times y''(0) + 1 = 1; \quad y''''(0) = 2 \times 0 \times 1 + 3 \times 0 = 0;$                          | M1         | 1.1b |
|  | $x = 0, y'''(0) = 1, y''''(0) = 0 \Rightarrow y''''''(0) = 5$   | A1         | 1.1b |
|  | $y = y(0) + y'(0)x + \frac{y''(0)}{2}x^2 + \frac{y'''(0)}{6}x^3 + \frac{y''''(0)}{24}x^4 + \frac{y''''''(0)}{120}x^5 + \dots$ | M1         | 2.5  |
|  | Series solution: $y = x + \frac{1}{6}x^3 + \frac{1}{24}x^5 + \dots$   | A1ft       | 1.1b |
|  | <b>(5)</b>  |            |      |
| <b>(9 marks)</b>   |   |            |      |
| <b>Notes:</b>  |   |            |      |
| <b>(a)</b>   |   |            |      |
| <b>M1:</b> Attempts to differentiate equation with use of the product rule   |   |            |      |
| <b>A1:</b> cao. Accept if terms all on one side  |   |            |      |
| <b>M1:</b> Continues the process of differentiating to progress towards the goal. Terms may be kept on one side, but an expression in the fourth derivative should be obtained |   |            |      |
| <b>A1:</b> Completes the process to reach the fifth derivative and rearranges to the correct form to obtain the correct answer by correct solution only                        |   |            |      |
| <b>(b)</b>   |   |            |      |
| <b>B1:</b> Deduces the correct value for $y''(0)$ from the information in the question   |   |            |      |
| <b>M1:</b> Finds the values of the derivatives at the given point  |   |            |      |
| <b>A1:</b> All correct   |   |            |      |
| <b>M1:</b> Correct mathematical language required with given denominators. Can be in factorial form  |   |            |      |
| <b>A1ft:</b> Correct series, must start $y = \dots$ . Follow through the values of their derivatives at 0  |   |            |      |

| Question | Scheme   | Marks      | AOs  |
|----------|--|------------|------|
| <b>5</b> | $y^2 = 4ax \Rightarrow 2y \frac{dy}{dx} = 4a$  | M1         | 2.1  |
|          | $\frac{dy}{dx} = \frac{2a}{y} \Rightarrow$ Gradient of normal is $\frac{-y}{2a} = -p$  | A1         | 1.1b |
|          | Equation of normal is : $y - 2ap = -p(x - ap^2)$   | M1         | 1.1b |
|          | Normal passes through $Q(aq^2, 2aq)$ so $2aq + apq^2 = 2ap + ap^3$   | M1         | 3.1a |
|          | Grad $OP \times$ Grad $OQ = -1 \Rightarrow \frac{2ap}{ap^2} \frac{2aq}{aq^2} = -1$   | M1         | 2.1  |
|          | $q = \frac{-4}{p}$   | A1         | 1.1b |
|          | $2a\left(\frac{-4}{p}\right) + ap\left(\frac{16}{p^2}\right) = 2ap + ap^3 \Rightarrow p^4 + 2p^2 - 8 = 0$  | M1         | 2.1  |
|          | $(p^2 - 2)(p^2 + 4) = 0 \Rightarrow p^2 = \dots$   | M1         | 1.1b |
|          | Hence (as $p^2 + 4 \neq 0$ ), $p^2 = 2^*$  | A1*        | 1.1b |
|          |  | <b>(9)</b> |      |
|          | <b>Alternative 1</b>   | M1         | 2.1  |
|          | First three marks as above and then as follows   | A1         | 1.1b |
|          |  | M1         | 1.1b |
|          | Solves $y^2 = 4ax$ and their normal simultaneously to find, in terms of $a$ and $p$ , either $x_Q \left( = ap^2 + 4a + \frac{4a}{p^2} \right)$ or $y_Q \left( = -2ap - \frac{4a}{p} \right)$ | M1         | 3.1a |
|          | Finds the second coordinate of $Q$ in terms of $a$ and $p$   | M1         | 1.1b |
|          | Both $x_Q = ap^2 + 4a + \frac{4a}{p^2}$ and $y_Q = -2ap - \frac{4a}{p}$  | A1         | 1.1b |
|          | Grad $OP \times$ Grad $OQ = -1 \Rightarrow \frac{2ap}{ap^2} \times \frac{-2ap - \frac{4a}{p}}{ap^2 + 4a + \frac{4a}{p^2}} = -1$  | M1         | 2.1  |
|          | Simplifies expression and solves: $4p^2 + 8 = p^4 + 4p^2 + 4$<br>$\Rightarrow p^4 - 4 = 0 \Rightarrow (p^2 - 2)(p^2 + 2) = 0 \Rightarrow p^2 = \dots$  | M1         | 2.1  |
|          | Hence (as $p^2 + 2 \neq 0$ ), $p^2 = 2^*$  | A1*        | 1.1b |
|          |  | <b>(9)</b> |      |

| Question   | Scheme  | Marks      | AOs  |
|--|---|------------|------|
| <b>5</b>   | <b>Alternative 2</b><br>First three marks as above and then as follows  | M1         | 2.1  |
|  |   | A1         | 1.1b |
|  |   | M1         | 1.1b |
|  | Solves $y^2 = 4ax$ and their normal simultaneously to find, in terms of $a$ and $p$ , either $x_Q \left( = ap^2 + 4a + \frac{4a}{p^2} \right)$ or $y_Q \left( = -2ap - \frac{4a}{p} \right)$  | M1         | 3.1a |
|  | Forms a relationship between $p$ and $q$ from their first coordinate:<br><b>either</b> $y_Q = 2a \left( -p - \frac{2}{p} \right) \Rightarrow q = -p - \frac{2}{p}$<br><b>or</b> $x_Q = a \left( p + \frac{2}{p} \right)^2 \Rightarrow q = \pm \left( p + \frac{2}{p} \right)$ | M1         | 2.1  |
|  | $q = -p - \frac{2}{p}$<br>(if $x$ coordinate used the correct root must be clearly identified before this mark is awarded)  | A1         | 1.1b |
|  | Grad $OP \times$ Grad $OQ = -1 \Rightarrow \frac{2ap}{ap^2} \times \frac{2aq}{aq^2} = -1 \left( \Rightarrow q = -\frac{4}{p} \right)$   | M1         | 2.1  |
|  | Sets $q = -p - \frac{2}{p} = -\frac{4}{p}$ and solves to give $p^2 = \dots$   | M1         | 1.1b |
|  | Hence $\left( \text{as } q = p + \frac{2}{p} = -\frac{4}{p} \text{ gives no solution} \right), p^2 = 2$ (only)*   | A1*        | 1.1b |
|  |   | <b>(9)</b> |      |
| <b>(9 marks)</b>   |   |            |      |
| <b>Notes:</b>  |   |            |      |
| <b>(a)</b>   |   |            |      |
| <b>M1:</b> Begins proof by differentiating and using the perpendicularity condition at point $P$ in order to find the equation of the normal |   |            |      |
| <b>A1:</b> Correct gradient of normal, $-p$ only   |   |            |      |
| <b>M1:</b> Use of $y - y_1 = m(x - x_1)$ . Accept use of $y = mx + c$ and then substitute to find $c$  |   |            |      |
| <b>M1:</b> Substitute coordinates of $Q$ into their equation to find an equation relating $p$ and $q$  |   |            |      |
| <b>M1:</b> Use of $m_1 m_2 = -1$ with $OP$ and $OQ$ to form a second equation relating $p$ and $q$   |   |            |      |
| <b>A1:</b> $q = \frac{-4}{p}$ only   |   |            |      |
| <b>M1:</b> Solves the simultaneous equations and cancels $a$ from their results to obtain a quadratic equation in $p^2$ only                 |   |            |      |
| <b>M1:</b> Attempts to solve their quadratic in $p^2$ . Usual rules  |   |            |      |
| <b>A1*:</b> Correct solution leading to given answer stated. No errors seen  |   |            |      |

**Question 5 notes continued:****Alternative 1:****M1A1M1:** As main scheme**M1:** Solves  $y^2 = 4ax$  and their normal simultaneously to find one of the coordinates for  $Q$  in terms of  $a$  and  $p$  as shown**M1:** Finds the second coordinate of  $Q$  in terms of  $a$  and  $p$ **A1:** Both coordinates correct in terms of  $a$  and  $p$ **M1:** Use of  $m_1m_2 = -1$  with  $OP$  and  $OQ$ . i.e.  $\frac{2ap}{ap^2} \times \frac{\text{their } y_Q}{\text{their } x_Q} = -1$  with coordinates of  $P$  and their expressions for  $x_Q$  and  $y_Q$ **M1:** Cancels the  $a$ 's, simplifies to a quadratic in  $p^2$  and solves the quadratic. Usual rules**A1\*:** Correct solution leading to the given answer stated. No errors seen**Alternative 2:****M1A1M1:** As main scheme**M1:** Solves  $y^2 = 4ax$  and their normal simultaneously to find one of the coordinates for  $Q$  in terms of  $a$  and  $p$  as shown**M1:** Uses their coordinate to form a relationship between  $p$  and  $q$ . Allow  $q = \left(p + \frac{2}{p}\right)$  for this mark**A1:** For  $q = -p - \frac{2}{p}$ . If the  $x$  coordinate was used to find  $q$  then consideration of the negative root is needed for this mark. Allow for  $q = \pm \left(p + \frac{2}{p}\right)$ **M1:** Use of  $m_1m_2 = -1$  with  $OP$  and  $OQ$  to form a second equation relating  $p$  and  $q$  only**M1:** Equates expressions for  $q$  and attempts to solve to give  $p^2 = \dots$ **A1\*:** Correct solution leading to the given answer stated. No errors seen. If  $x$  coordinate used, invalid solution must be rejected



| Question          | Scheme   | Marks | AOs  |
|-------------------|--|-------|------|
| <b>6(a)</b>       | $\mathbf{AB} \times \mathbf{AC} = \begin{pmatrix} -1 \\ -1 \\ -1 \end{pmatrix} \times \begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix} = \begin{pmatrix} -2-1 \\ -1+2 \\ 1+1 \end{pmatrix} = \begin{pmatrix} -3 \\ 1 \\ 2 \end{pmatrix}$ | M1    | 1.1b |
|                   | $\mathbf{r} \cdot \begin{pmatrix} -3 \\ 1 \\ 2 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \cdot \begin{pmatrix} -3 \\ 1 \\ 2 \end{pmatrix} = 1$   | M1    | 1.1b |
|                   | Hence $-3x + y + 2z = 1$   | A1    | 1.1b |
|                   | <b>(3)</b>   |       |      |
| <b>(b)</b>        | Volume of Tetrahedron = $\frac{1}{6}  \mathbf{n} \cdot (\mathbf{AD}) $   | M1    | 3.1a |
|                   | $= \frac{1}{6} \left  \begin{pmatrix} -3 \\ 1 \\ 2 \end{pmatrix} \cdot \left( \begin{pmatrix} 10 \\ 5 \\ 5 \end{pmatrix} - \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix} \right) \right $  | M1    | 1.1b |
|                   | $= \frac{1}{6}  (-27 + 3 + 8)  = \frac{8}{3}$  | A1    | 1.1b |
|                   | <b>(3)</b>   |       |      |
| <b>(c)</b>        | $\mathbf{AE} = k\mathbf{AC}$ so $E$ is $(1+k, 2-k, 1+2k)$  | M1    | 3.1a |
|                   | $E$ lies on plane so $2(1+k) - 3(2-k) + 3 = 0$ , leading to $k = \dots$  | M1    | 3.1a |
|                   | Hence $k = \frac{1}{5}$  | A1    | 1.1b |
|                   | <b>(3)</b>   |       |      |
| <b>(d)</b>        | Volume $ABEF = \frac{1}{6} (\mathbf{AB} \times \mathbf{AE}) \cdot \mathbf{AF} = \frac{1}{6} \left( \mathbf{AB} \times \frac{1}{5} \mathbf{AC} \right) \cdot \frac{1}{9} \mathbf{AD}$   | M1    | 3.1a |
|                   | $= \frac{1}{45} \left( \frac{1}{6} (\mathbf{AB} \times \mathbf{AC}) \cdot \mathbf{AD} \right)$ and hence result *  | A1*   | 2.2a |
|                   | <b>(2)</b>   |       |      |
| <b>(11 marks)</b> |  |       |      |

| <b>Question 6 notes:</b> |   |
|--------------------------|---|
| <b>(a)</b>               | <p><b>M1:</b> Attempting a suitable cross product. Accept use of unit vectors</p> <p><b>M1:</b> Complete method that would lead to finding the Cartesian equation of plane</p> <p><b>A1:</b> Accept any equivalent form</p>   |
| <b>(b)</b>               | <p><b>M1:</b> Identifies suitable vectors and attempts to substitute into a correct formula. Accept use of unit vectors</p> <p><b>M1:</b> Correct form of scalar triple product using their <b>n</b> from part (a)</p> <p><b>A1:</b> <math>\frac{8}{3}</math> or exact equivalent form</p>            |
| <b>(c)</b>               | <p><b>M1:</b> Uses that <math>E</math> is on <math>AC</math> in order to find an expression for <math>E</math></p> <p><b>M1:</b> Uses that <math>E</math> is in the plane <math>\Pi</math> to form and solve an expression in <math>k</math></p> <p><b>A1:</b> <math>\frac{1}{5}</math> o.e. only</p> |
| <b>(d)</b>               | <p><b>M1:</b> Uses formula for volume of tetrahedron and substitutes for <b>AE</b> and <b>AF</b></p> <p><b>A1*:</b> Deduces result: Use of <math>\frac{1}{6}(\mathbf{AB} \times \mathbf{AC}) \cdot \mathbf{AD}</math> is required and no errors seen in solution</p>                                  |

| Question         | Scheme  | Marks | AOs  |
|------------------|---|-------|------|
| 7                | $x^2 + 4y^2 = 4 \Rightarrow 2x + 8y \frac{dy}{dx} = 0 \Rightarrow \frac{dy}{dx} = \dots$  | M1    | 3.1a |
|                  | Equation of tangent at $P(x_1, y_1)$ is $(y - y_1) = -\frac{x_1}{4y_1}(x - x_1)$  | M1    | 3.1a |
|                  | $xx_1 + 4yy_1 = x_1^2 + 4y_1^2 = 4$ and at $Q(x_2, y_2)$ : $xx_2 + 4yy_2 = 4$   | A1    | 2.2a |
|                  | Intersect at $(r, s)$ gives $rx_1 + 4sy_1 = 4$ and $rx_2 + 4sy_2 = 4$   | B1    | 2.1  |
|                  | Uses their previous results to find the gradient of the line $l$  | M1    | 3.1a |
|                  | $\frac{y_2 - y_1}{x_2 - x_1} = \frac{-r}{4s}$   | A1    | 1.1b |
|                  | Equation of $l$ is $y - y_1 = \frac{-r}{4s}(x - x_1)$   | M1    | 2.1  |
|                  | $4sy + rx = 4sy_1 + rx_1 = 4^*$   | A1*   | 2.2a |
|                  | <b>(8)</b>  |       |      |
| <b>(8 marks)</b> |   |       |      |
| <b>Notes:</b>    |   |       |      |
| <b>M1:</b>       | Attempts to solve the problem by using differentiation to obtain an expression for $\frac{dy}{dx}$  |       |      |
| <b>M1:</b>       | Realise the need to form a general equation of the tangent at $(x_1, y_1)$ . May use alternative variables  |       |      |
| <b>A1:</b>       | Deduces $x_1^2 + 4y_1^2 = 4$ to obtain a correct equation and deduces a correct second equation   |       |      |
| <b>B1:</b>       | Uses $(r, s)$ in both equations to form the two given equations or exact equivalents  |       |      |
| <b>M1:</b>       | Uses their previous results to find the gradient of the line $l$  |       |      |
| <b>A1:</b>       | $\frac{-r}{4s}$   |       |      |
| <b>M1:</b>       | Formulates the line $l$ with their $\frac{-r}{4s}$ . Use of $y - y_1 = m(x - x_1)$ or $y = mx + c$ with their gradient and an attempt to find $C$ |       |      |
| <b>A1*:</b>      | Correct solution leading to $4sy + rx = 4sy_1 + rx_1$ with deduction that this equals 4 as $(x_1, y_1)$ is on the ellipse. No errors seen         |       |      |

| Question                | Scheme   | Marks      | AOs  |
|-------------------------|--|------------|------|
| <b>8(a)</b>             | $h(x) = 45 + 15 \sin x + 21 \sin\left(\frac{x}{2}\right) + 25 \cos\left(\frac{x}{2}\right)$  |            |      |
|                         | $\frac{dh}{dx} = 15 \cos x + \frac{21}{2} \cos\left(\frac{x}{2}\right) - \frac{25}{2} \sin\left(\frac{x}{2}\right)$  | M1         | 1.1b |
|                         | $\frac{dh}{dx} = \dots + \dots \frac{1-t^2}{1+t^2} - \dots \frac{2t}{1+t^2}$   | M1         | 1.1a |
|                         | e.g. $\frac{dh}{dx} = \dots \left( 2 \left( \frac{1-t^2}{1+t^2} \right)^2 - 1 \right) + \dots$ <b>or</b> $\frac{dh}{dx} = \dots \frac{1 - \left( \frac{2t}{1-t^2} \right)^2}{1 + \left( \frac{2t}{1-t^2} \right)^2} + \dots$               | M1         | 3.1a |
|                         | e.g. $\frac{dh}{dx} = 15 \left( 2 \left( \frac{1-t^2}{1+t^2} \right)^2 - 1 \right) + \frac{21}{2} \left( \frac{1-t^2}{1+t^2} \right) - \frac{25}{2} \left( \frac{2t}{1+t^2} \right)$   | A1         | 1.1b |
|                         | $\dots = \frac{15[4(1-t^2)^2 - 2(1+t^2)^2] + 21(1-t^2)(1+t^2) - 50t(1+t^2)}{2(1+t^2)^2} x$   | M1         | 2.1  |
|                         | $\dots = \frac{9t^4 - 50t^3 - 180t^2 - 50t + 51}{2(1+t^2)^2} = \frac{(t^2 - 6t - 17)(9t^2 + 4t - 3)}{2(1+t^2)^2} *$  | A1*        | 2.1  |
|                         |  | <b>(6)</b> |      |
| <b>8(a) Alternative</b> | $h(x) = \dots + 21 \left( \frac{2t}{1+t^2} \right) + 25 \left( \frac{1-t^2}{1+t^2} \right)$  | M1         | 1.1a |
|                         | $= \dots + 15 \left[ 2 \left( \frac{2t}{1+t^2} \right) \left( \frac{1-t^2}{1+t^2} \right) \right] + \dots$ <b>or</b> $= \dots + 15 \left( \frac{2 \left( \frac{2t}{1-t^2} \right)}{1 + \left( \frac{2t}{1-t^2} \right)^2} \right) + \dots$ | M1         | 2.1  |
|                         | $h(x) = 45 + \frac{15(4t(1-t^2)) + 42t(1+t^2) + 25(1-t^4)}{(1+t^2)^2}$   | M1         | 1.1b |
|                         | $h(x) = 45 - \frac{25t^4 + 18t^3 - 102t - 25}{(1+t^2)^2}$ <b>or</b> $\frac{20t^4 - 18t^3 + 90t^2 + 102t + 70}{(1+t^2)^2}$  | A1         | 1.1b |
|                         | $\frac{dh}{dx} = \frac{dh}{dt} \times \frac{dt}{dx} = \frac{('u')(1+t^2)^2 - ('u')(4t(1+t^2))}{(1+t^2)^4} \times \frac{1}{4}(1+t^2)$   | M1         | 3.1a |
|                         | $\dots = \frac{9t^4 - 50t^3 - 180t^2 - 50t + 51}{2(1+t^2)^2} = \frac{(t^2 - 6t - 17)(9t^2 + 4t - 3)}{2(1+t^2)^2} *$  | A1*        | 2.1  |
|                         |  | <b>(6)</b> |      |

| Question   | Scheme   | Marks      | AOs  |
|--|--|------------|------|
| <b>8(b)(i)</b>   | Accept any value between $\frac{1}{40} = 0.025$ and $\frac{1}{60} \approx 0.167$ inclusive   | B1         | 3.3  |
| <b>(ii)</b>  | Suitable for times since the graphs both oscillate bi-modally with about the same periodicity  | B1         | 3.4  |
|  | Not suitable for predicting heights since the heights of the peaks vary over time, but the graph of $h(x)$ has fixed peak height   | B1         | 3.5b |
|  |  | <b>(3)</b> |      |
| <b>8(c)</b>  | Solves at least one of the quadratics<br>$t = \frac{6 \pm \sqrt{36 - 4 \times 1 \times 17}}{2} = 3 \pm \sqrt{26}$ or $t = \frac{-4 \pm \sqrt{16 - 4 \times 9 \times (-3)}}{18} = \frac{-2 \pm \sqrt{31}}{9}$ | <b>M1</b>  | 1.1b |
|  | Finds corresponding $x$ values, $x = 4 \tan^{-1}(t)$ for at least one value of $t$ from the $9t^2 + 4t - 3$ factor   | <b>M1</b>  | 1.1b |
|  | One correct value for these $x$<br>e.g. $x = \arctan -2.797$ or $9.770, 1.510$   | <b>A1</b>  | 1.1b |
|  | Maximum peak height occurs at smallest positive value of $x$ , from first graph, but the third of these peaks needed,<br>So $t = 1.509... + 8\pi = 26.642$ is the required time                              | <b>M1</b>  | 3.4  |
|  | $x = 26.642$ corresponds to 26 hours and 39 minutes (nearest minute) after 08:00 on 3rd January<br>(Allow if a different greatest peak height used)  | <b>M1</b>  | 3.4  |
|  | Time of greatest tide height is approximately 10:39 (am)<br>(also allow 10:38 or 10:40)  | <b>A1</b>  | 3.2a |
|  |  | <b>(6)</b> |      |
| <b>(15 marks)</b>  |  |            |      |
| <b>Notes:</b>  |  |            |      |
| <b>(a)</b>   |  |            |      |
| <b>M1:</b> Differentiates $h(x)$   |  |            |      |
| <b>M1:</b> Applies $t$ -substitution to both $\left(\frac{x}{2}\right)$ terms with their coefficients  |  |            |      |
| <b>M1:</b> Forms a correct expression in $t$ for the $\cos x$ term, using double angle formula and $t$ -substitution, or double ' $t$ '-substitution |  |            |      |
| <b>A1:</b> Fully correct expression in $t$ for $\frac{dh}{dx}$   |  |            |      |
| <b>M1:</b> Gets all terms over the correct common factor. Numerators must be appropriate for their terms   |  |            |      |
| <b>A1*:</b> Achieves the correct answer via expression with correct quartic numerator before factorisation   |  |            |      |

**Question 8 notes continued:****Alternative:****(a)****M1:** Applies  $t$ -substitution to both  $\left(\frac{x}{2}\right)$  terms**M1:** Forms a correct expression in  $t$  for the  $\sin x$  term, using double angle formula and  $t$ -substitution, or double ' $t$ '-substitution**M1:** Gets all terms in  $t$  over the correct common factor. Numerators must be appropriate for their terms. May include the constant term too**A1:** Fully correct expression in  $t$  for  $h(x)$ **M1:** Differentiates, using both chain rule and quotient rule with their ' $u$ '**A1\*:** Achieves the correct answer via expression with correct quartic numerator before factorisation**Note:** The individual terms may be differentiated before putting over a common denominator. In this case score the third M for differentiating with chain rule and quotient rule, then r return to the original scheme**(b)(i)****B1:** Any value between  $\frac{1}{40}$  (e.g. taking  $h(0)$  as reference point) or  $\frac{1}{60}$  (taking lower peaks as reference)**NB:** Taking high peak as reference gives  $\frac{1}{50}$ **(b)(ii)****B1:** Should mention both the bimodal nature and periodicity for the actual data match the graph of  $h$ **B1:** Mentions that the heights of peaks vary in each oscillation**(c)****M1:** Solves (at least) one of the quadratic equations in the numerator**M1:** Must be attempting to solve the quadratic factor from which the solution comes $9t^2 + 4t - 3$  and using  $t = \tan\left(\frac{x}{4}\right)$  to find a corresponding value for  $x$ **A1:** At least one correct  $x$  value from solving the requisite quadratic: awrt any of  $-2.797$ ,  $1.510$ ,  $9.770$ ,  $14.076$ ,  $22.336$ ,  $26.642$ ,  $34.902$  or  $39.208$ **M1:** Uses graph of  $h$  to pick out their  $x = 26.642$  as the time corresponding to the third of the higher peaks, which is the highest of the peaks on 4th January on the tide height graph. As per scheme or allow if all times listed and correct one picked**M1:** Finds the time for one of the values of  $t$  corresponding to the highest peaks. E.g.  $1.5096\dots \sim 09:31$  (3rd January) or  $14.076\dots \sim 22:05$  (3rd January) or  $26.642\dots \sim 10:39$  (4th January) or  $39.208\dots \sim 23:13$  (4th January). (Only follow through on use of the smallest positive  $t$  solution  $+ 4k\pi$ )**A1:** Time of greatest tide height on 4th January is approximately 10:39. Also allow 10:38 or 10:40